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1. Introduction

Magnetic resonance imaging (MRI) is an important diagnostic tool in medical imaging, enabling specialists to visualize internal body structures non-invasively. By taking precise measurements over time, MRI can be used to infer dynamic information through the (re)construction of deformation fields. These deformation fields are essential for a-posteriori analysis, serving as the starting point for inferring other mechanical parameters such as stress and strain on the body.

Raw MR measurements, commonly referred to as k-space data, are acquired in the spatial frequency domain. Typical methods for inferring deformation fields and other dynamical information, however, are based on analysis in the spatial domain. Acquiring k-space data with sufficiently high resolution in both the temporal and spatial domain can be costly. Therefore, we aim to infer dynamical information, specifically the deformation field, directly from raw MRI measurements in the spatial frequency domain, see Figure 1.

2. Problem statement

We consider a scenario where it is reasonable to assume that the amount of material (tissue) in a region can only change due to fluxes on the boundary. This naturally leads to the assumption that the magnetization m is conserved, and thus satisfies the continuity equation, i.e.,

$$\frac{\partial m}{\partial t}(t,x) + \operatorname{div}(m\boldsymbol{v})(t,x) = 0, \quad (t,x) \in (0,\tau) \times \mathcal{R}. \tag{1}$$

Here $\tau > 0$ is an integration time, $\mathcal{R} \subset \mathbb{R}^n$ is a rectangular domain in either two or three dimensions, and $\boldsymbol{v}$ is the (Eulerian) velocity field underlying the motion. Our objective is to infer a deformation field $\boldsymbol{u}$ from a finite number of k-space measurements over time by exploiting the conservation law in the spatial frequency domain, which is given by

$$\frac{\partial \widehat{m}}{\partial t}(t,\xi) = -2\pi i \sum_{j=1}^{n} \xi_j \left(\widehat{m} * \widehat{v}_j\right)(t,\xi), \quad (t,\xi) \in (0,\tau) \times \mathbb{R}^n. \tag{2}$$

The latter equation establishes a direct relationship between the Fourier Transform $\widehat{m}$ of the magnetization m (the k-space data) and the Fourier Transform $\widehat{\boldsymbol{v}}$ of the velocity field. Here and in what follows the Fourier Transform always refers to the Fourier Transform in space. In practice, we consider (2) on a rectangular subdomain $\Xi \subset \mathbb{R}^n$, as the Fourier Transform $|\widehat{m}|$ will be negligibly small sufficiently far away from the origin.

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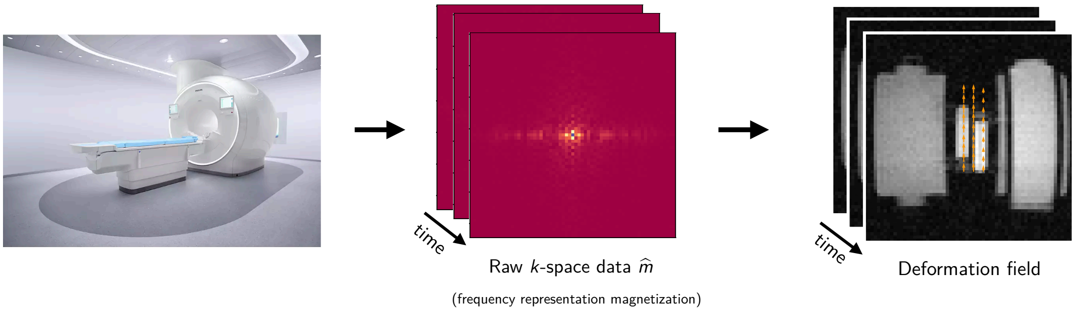

FIGURE 1. High level overview of the goal of this project: infer a deformation field directly from k-space data.

If we are able to reconstruct $\widehat{\boldsymbol{v}}$ from data, e.g., by exploiting (2), we may subsequently reconstruct $\widehat{\boldsymbol{u}}$ using the frequency representation of the material derivative:

$$
\begin{cases} \dfrac{\partial \widehat{u}_l}{\partial t} = \widehat{v}_l - \displaystyle\sum_{j=1}^{n} \mathcal{D}_j(\widehat{u}_l) * \widehat{v}_j, & (t,\xi) \in (0,\tau) \times \Xi, \\ \widehat{\boldsymbol{u}}(0,\xi) = \mathbf{0} & \xi \in \Xi, \end{cases} \qquad 1 \le l \le n. \tag{3}
$$

Here we have abbreviated the frequency representation of $\frac{\partial}{\partial x_j}$ by $\mathcal{D}_j$, i.e., $\mathcal{D}_j g(\xi) = 2\pi i \xi_j g(\xi)$ for sufficiently regular functions $g : \mathbb{R}^n \to \mathbb{C}$. The initial condition for the deformation field corresponds to using the configuration at $t = 0$ as reference domain.

Problem statement *Assume we are given a finite set of k-space measurements $\mathscr{D} \subset [0,\tau] \times \Xi$, e.g., uniformly sampled in the time and spatial frequency domain, and that the magnetization and velocity field are related by* (1)*. Reconstruct the velocity field $\boldsymbol{v}$, deformation field $\boldsymbol{u}$ and magnetization m on $[0,\tau] \times \mathcal{R}$.*

3. Challenges

Mathematically, the problem boils down to solving an inverse problem for the continuity equation in the frequency domain (2). Specifically, we treat the velocity field $\widehat{\boldsymbol{v}}$ as a parameter to be reconstructed, along with the associated deformation field $\widehat{\boldsymbol{u}}$, from partial observations in the frequency domain. Since (2) is essentially an infinite-dimensional ODE, the inverse problem can be naturally framed as an optimal control problem: find $(\widehat{m}_0, \widehat{\boldsymbol{v}})$ such that the solution of

$$
\begin{cases} \dfrac{\partial \widehat{m}}{\partial t}(t,\xi) = -2\pi i \displaystyle\sum_{j=1}^{n} \xi_j \left(\widehat{m} * \widehat{v}_j\right)(t,\xi), & (t,\xi) \in (0,\tau) \times \Xi, \\ \widehat{m}(0,\xi) = \widehat{m}_0(\xi), \end{cases} \tag{4}
$$

"agrees" with the measured k-space data $\mathscr{D}$. Here $\widehat{m}_0 : \Xi \to \mathbb{C}$ is an initial condition for the Fourier Transform of the magnetization, and $\widehat{\boldsymbol{v}}$ the unknown frequency representation of the velocity field.

More formally, we introduce a loss $L(\widehat{m}\,(\cdot;\widehat{m}_0;\widehat{\boldsymbol{v}})\,;\mathscr{D})$, measuring the discrepancy between the solution $\widehat{m} = \widehat{m}\,(\cdot;\widehat{m}_0;\widehat{\boldsymbol{v}})$ of (4) and the data $\mathscr{D}$ for a given estimate of $(\widehat{m}_0, \widehat{\boldsymbol{v}})$, and solve the optimal control problem

$$
\begin{cases} \displaystyle\min_{(\widehat{m}_0,\widehat{\boldsymbol{v}})} L(\widehat{m}\,(\cdot;\widehat{m}_0;\widehat{\boldsymbol{v}})\,;\mathscr{D}), \\ \text{s.t. } \dfrac{\partial \widehat{m}}{\partial t}(t,\xi) = -2\pi i \displaystyle\sum_{j=1}^{n} \xi_j \left(\widehat{m} * \widehat{v}_j\right)(t,\xi), & (t,\xi) \in (0,\tau) \times \Xi, \\ \qquad \widehat{m}(0,\xi) = \widehat{m}_0(\xi), & \xi \in \Xi. \end{cases}
$$

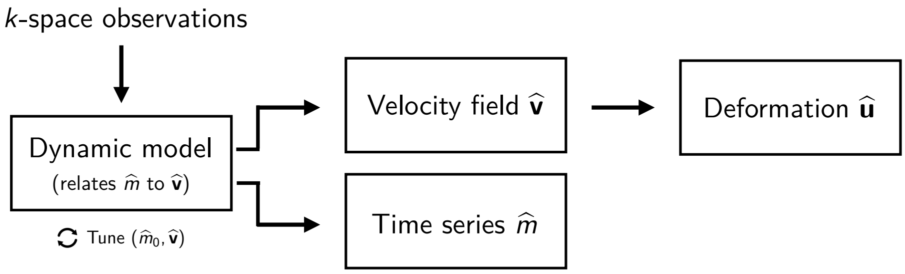

FIGURE 2. Overview of the setup. We tune the initial condition $\widehat{m}_0$ and frequency representation of the velocity field $\widehat{\boldsymbol{v}}$ to match the dynamical model (4) with the observed k-space measurements. After optimization, we use the dynamical model to reconstruct $\widehat{m}$. Furthermore, we reconstruct the frequency representation $\widehat{\boldsymbol{u}}$ of the deformation field by integrating (3).

Afterwards, we reconstruct the frequency representation $\widehat{\boldsymbol{u}}$ of the deformation field by integrating (3). See Figure 2 for an overview of the setup.

We are looking for mathematicians with expertise in inverse problems for ODEs and PDEs, optimal control theory, numerical methods for stiff ODEs and approximation theory, who can help us address the following challenges.

Question 3.1 (Measuring discrepancies)**.** How should we measure discrepancies between observations and predictions in k-space? When we naively use an unweighted L^2-loss, we typically encounter issues, and get stuck in bad local minima. We hypothesize that this is related to the relatively fast decay of the k-space data as we move away from the origin. Should we rescale the k-space data to make the optimization problem more easily solvable? If so, how? Or should we instead incorporate manually crafted weights into the L^2-loss? Or should we perhaps use an entirely different notion of discrepancy altogether?

Question 3.2 (Regularization velocity)**.** The optimal control problem is severely ill-posed. We therefore need to regularize $\widehat{\boldsymbol{v}}$. What kind of regularization should we use to stabilize the optimization problem? A good choice for regularization is crucial, since it heavily influences the IVP in (3) and thus the reconstruction of $\widehat{\boldsymbol{u}}$.

Question 3.3 (Regularization deformation field)**.** In applications, the inferred deformation field $\boldsymbol{u}$ is often used to compute bio-mechanical parameters such as strain and stress. This typically involves evaluating the spatial derivatives of $\boldsymbol{u}$. If our reconstruction of $\widehat{\boldsymbol{u}}$ is noisy, however, we will not be able to estimate the needed spatial derivatives with sufficient accuracy. Is there a computationally efficient way to control the behavior of the spatial derivatives of $\boldsymbol{u}$ during optimization?

Question 3.4 (Representation velocity field)**.** What are efficient numerical representations for $\widehat{\boldsymbol{v}}$ and $\widehat{m}_0$? Should we use discretized representations on $[0,\tau] \times \Xi$? Or should we use representations that can be evaluated on all of $[0,\tau] \times \Xi$, such as splines or orthogonal series expansions, enabling more memory-efficient implementations? Alternatively, we could parameterize $\boldsymbol{v}$, or even $\boldsymbol{u}$, in the spatial domain and *compute* $\widehat{\boldsymbol{v}}$; is there a preferred choice?

Question 3.5 (Stiff ODEs)**.** The ODEs in (2) and (3) are stiff essentially due to the multiplications by ξ in the frequency domain. Are there ways to "precondition" the ODEs, so to speak, to remedy this issue (at least partially)? If the problem was autonomous, for instance, one could think about using variation of constants type of arguments. Is there something analogous to consider here in the non-autonomous case?

References

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